Exercise 9

Show that

$$\frac{d}{dx}\left(\ln x + \frac{\ln x}{2} + \frac{\ln x}{4} + \cdots\right) = \frac{2}{x}, \ x > 0$$

Solution

Inspecting the series in the parentheses, we see that it is geometric. The first term is

$$a_1 = \ln x,$$

and the common ratio is

$$r = \frac{1}{2}.$$

Consequently, the sum of the series is

$$S = \frac{a_1}{1 - r}$$
$$= \frac{\ln x}{1 - \frac{1}{2}}$$
$$= \frac{\ln x}{\frac{1}{2}}$$
$$= 2\ln x.$$

Differentiating $2\ln x$ gives us $2/x$, the desired result. Therefore,

$$\frac{d}{dx}\left(\ln x + \frac{\ln x}{2} + \frac{\ln x}{4} + \cdots\right) = \frac{2}{x}, \ x > 0.$$